MID-SEMESTER EXAM ISI BANGALORE 11TH MARCH 2013

REPRESENTATION THEORY OF FINITE GROUPS 100 POINTS

Notes.

(a) Leave sufficient gap between your answers to different questions. Preferably, begin each answer on a separate page.

(b) Justify all your steps. Assume only those results that have been proved in class. All other steps should be justified.

(c) $\mathbb{Z}$ = integers, $\mathbb{Q}$ = rational numbers, $\mathbb{R}$ = real numbers, $\mathbb{C}$ = complex numbers.

(d) Generic notation: G is a group, k is a field, V is a G-module over k.

---

1. [5 points] Let $A = (a_{ij})$ be an $n \times n$ matrix over a field k such that A is strictly upper triangular, i.e., $a_{ij} = 0$ for $i \geq j$. Describe the eigenvalues of A and prove that A is diagonalizable iff A is the zero matrix. (Use elementary linear algebra only.)

2. [11 points] List all similarity classes of 5×5 matrices A over $\mathbb{C}$ such that A satisfies the following two conditions.
   (a) The characteristic polynomial $p(x)$ of A is $(x + 4)^5$.
   (b) $(A + 4I)^2 \neq 0$.

3. [9 points] Give an example of a G-module V over a field k and a subspace $W \subset V$ such that W is not a submodule of V.

4. [11 points] Let $G = C_2 = \{1, \tau \mid \tau^2 = 1\}$ be the cyclic group of order 2. Consider the action of G on $V = \mathbb{R}^3$ given by

$$\rho(\tau) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Decompose V as a direct sum of irreducible G-modules.

5. [9 points] Give an example of a irreducible G-module V over k such that the dimension of V over k is greater than 1.

6. [9 points] Give an example of a G-module V and a submodule $W \subset V$ such that W has no G complement, i.e., there is no submodule $W' \subset V$ for which $V = W \oplus W'$ holds.

7. [7 points] Prove that $\mathbb{Q} \otimes_{\mathbb{Z}} M = 0$ for any finite abelian group M.

8. [9 points] Let $V = k^3$. Give an example of an element in $V \otimes_k V$ that is not a pure tensor, i.e., it is not of the form $v \otimes v'$ for any $v, v' \in V$. Justify your answer.

9. [12 points] Let R be a commutative ring and M an R-module with generators $\{m_i\}_{i \in I}$. Expand the following products and express them as a sum of homogeneous terms of different degrees.

(i) $x \otimes x \in T(M)$ where $x = m_1 + (m_2 \otimes m_3) + (m_1 \otimes m_4 \otimes m_5) \in T(M)$.
(ii) $x \cdot x \in S(M)$ where $x = m_1 + (m_2 \cdot m_3) + (m_1 \cdot m_4 \cdot m_5) \in S(M)$.
(iii) $x \wedge x \in \bigwedge(M)$ where $x = m_1 + (m_2 \wedge m_3) + (m_1 \wedge m_4 \wedge m_5) \in \bigwedge(M)$.

10. [18 points] Let $G = S_3$ be the permutation group on 3 symbols and let H be the normal subgroup of order 3. Find a one-dimensional H-module $\mathbb{L}$ (over $\mathbb{C}$) such that the induced module $\mathrm{Ind}_H^G(\mathbb{L})$ is isomorphic to the standard module for S_3. (You must provide the isomorphism also.)